Recall: inner product space $V, \langle \cdot , \cdot \rangle$

orthonormal basis β

Goal: Study maps between inner product space.

$$(V, \langle \, , \, \rangle_V, \beta) \xrightarrow{f} (W, \langle \, , \, \rangle_W, \gamma)$$

§ Adjoint of a linear operator

<u>Theorem</u>: (Representation of linear functionals)

Let V be a finite-dim inner product space F

Then for any linear transformation $g: V \longrightarrow F$

there exists unique $\vec{y} \in V$. s.t. $g(\vec{x}) = \langle \vec{x}, \vec{y} \rangle \quad \forall \vec{x} \in V$

Def: linear functional

pf: Let $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ be an orthonormal basis for V.

- Uniqueness: $g(\vec{v}_i) = \langle \vec{v}_i, \vec{y} \rangle$

$$\Rightarrow \vec{y} = \sum_{i=1}^{n} \langle \vec{y}, \vec{v}_i \rangle \vec{v}_i = \sum_{i=1}^{n} \overline{g(\vec{v}_i)} \cdot \vec{v}_i .$$

- Let $\vec{y} = \sum_{i=1}^{n} \overline{g(\vec{v}_i)}\, \vec{v}_i$

then for $\vec{x} = \vec{v}_j$, $\langle \vec{v}_j, \vec{y} \rangle = \sum_{i=1}^{n} \langle \vec{v}_j, \overline{g(\vec{v}_i)}\, \vec{v}_i \rangle = g(\vec{v}_j)$

By linearity of g and $\langle \cdot, \vec{y} \rangle$, $g(\vec{x}) = \langle \vec{x}, \vec{y} \rangle \ \forall \vec{x} \in V$.

$\square$

Theorem: Let V be fin-dim inner product space, and let T be a linear operator on V. Then there exits a unique linear operator $T^*: \longrightarrow V$. s.t. $\boxed{\langle T(\vec{x}), \vec{y} \rangle = \langle \vec{x}, T^*(\vec{y}) \rangle}$ $\forall \vec{x}, \vec{y} \in V$

Def: T^* is called the adjoint of T.

Pf: Given any $\vec{y} \in V$. then the map $g_{\vec{y}}: V \longrightarrow F$

$\vec{x} \rightsquigarrow \langle T(\vec{x}), \vec{y} \rangle$

is linear since T is linear and $\langle \cdot, \vec{y} \rangle$ is linear.

By the previous representation theorem, there exists unique $\vec{y}' \in V$.

s.t. $\quad \langle T(\vec{x}), \vec{y} \rangle \overset{= g_{\vec{y}}(\vec{x})}{=} \langle \vec{x}, \vec{y}' \rangle$ $\quad =: T^*(\vec{y})$

This uniquely defines a map $T^* : V \longrightarrow V$ by $T^*(\vec{y}) = \vec{y}'$

- To see that T^* is linear, let $\vec{y}_1, \vec{y}_2 \in V$ and $c \in F$. Then $\forall \vec{x} \in V$

$$
\begin{aligned}
\langle \vec{x}, T^*(\vec{y}_1 + \vec{y}_2) \rangle &= \langle T(\vec{x}), \vec{y}_1 + \vec{y}_2 \rangle = \langle T(\vec{x}), \vec{y}_1 \rangle + \langle T(\vec{x}), \vec{y}_2 \rangle \\
&= \langle \vec{x}, T^*(\vec{y}_1) \rangle + \langle \vec{x}, T^*(\vec{y}_2) \rangle \\
&= \langle \vec{x}, T^*(\vec{y}_1) + T^*(\vec{y}_2) \rangle
\end{aligned}
$$

Similarly, prove $T^*(c\vec{y}) = c\, T^*(\vec{y})$ $\quad \square$

Rmk:

(1) $\langle \vec{x}, T(\vec{y}) \rangle = \overline{\langle T(\vec{y}), \vec{x} \rangle} = \overline{\langle \vec{y}, T^*(\vec{x}) \rangle} = \langle T^*(\vec{x}), \vec{y} \rangle$

(2) More generally, if $T: V \to W$, can define $T^*: W \longrightarrow V$
$\vec{x} \rightsquigarrow T(\vec{x})$. $\quad \vec{y} \rightsquigarrow T^*(y)$

s.t. $\boxed{\langle T(\vec{x}), \vec{y} \rangle_W = \langle \vec{x}, T^*(\vec{y}) \rangle_V}$.

(3). Adjoint may not exist for ∞-dim vector space.

Recall: $A \in M_{n\times n}(F)$, then $A^* := \overline{A^T}$ conjugate transpose / adjoint

Prop: Let V be a finite-dim inner product space, and let β be an orthonormal basis for V. Then $\forall T \in \mathcal{L}(V)$,

we have $[T^*]_\beta = [T]_\beta^*$ ← conjugate transpose of matrix.

adjoint of T

Pf: Let $A = [T]_\beta$. $B = [T^*]_\beta$, and $\beta = \{\vec{v_1}, \cdots, \vec{v_n}\}$.

Then $B_{ij} = \langle T^*(\vec{v_j}), \vec{v_i} \rangle = \langle v_j, T(\vec{v_i}) \rangle = \overline{\langle T(\vec{v_i}), \vec{v_j} \rangle}$

$= \overline{A_{ji}}$ $\square$

Adjoint. $T: V \to V.$ $\xleftrightarrow[\ [T^*]_\beta = A^*\ ]{[T]_\beta = A}$ $A \in M_{n\times n}(F)$

$\rightsquigarrow$ $T^*: V \to V.$ $\rightsquigarrow$ $A^* = \overline{A}^T$

β orthonormal basis

s.t. $\langle T(\vec{x}), \vec{y} \rangle = \langle \vec{x}, T^*(\vec{y}) \rangle$ $(A\vec{x})^T \cdot \vec{y} = \vec{x}^T \cdot (A^*\vec{y})$

Prop:

Equivalently, same properties for matrices

(a) $(T+U)^* = T^* + U^*$

(b) $(cT)^* = \bar{c} \cdot T^*$. $\forall c \in F$

(c) $(TU)^* = U^* T^*$

(d) $T^{**} = T$

(e) $I^* = I$

(a) $(A+B)^* = A^* + B^*$

(b) $(c \cdot A)^* = \bar{c} \cdot A^*$ $\forall c \in F$

(c) $(AB)^* = B^* A^*$

(d) $A^{**} = A$

(e) $(I_n)^* = I_n$

Pf of (a): $\langle \vec{x}, (T+U)^*\vec{y} \rangle = \langle (T+U)\vec{x}, \vec{y} \rangle$

$\forall \vec{x}, \vec{y} \in V$

$$= \langle T(\vec{x}), \vec{y} \rangle + \langle U(\vec{x}), \vec{y} \rangle$$
$$= \langle \vec{x}, T^*(y) \rangle + \langle \vec{x}, U^*(\vec{y}) \rangle$$
$$= \langle \vec{x}, (T^*+U^*)(\vec{y}) \rangle.$$

Hence, $(T+U)^* = T^* + U^*$

Pf of (d). $\langle \vec{x}, T^{**}(\vec{y}) \rangle = \langle T^*(\vec{x}), \vec{y} \rangle$

$$= \langle \vec{x}, T(\vec{y}) \rangle$$

Hence $T^{**} = T$.

$\square$

Prop: Suppose $T \in \mathcal{L}(V)$ has an eigenvalue λ.
then T^* has an eigenvalue $\bar{\lambda}$.

(Equivalently, Suppose $A \in M_{n\times n}(F)$ has an eigenvalue λ,
then A^* has an eigenvalue $\bar{\lambda}$.)

Note: the eigenvectors of T and T^* w.r.t λ and $\bar{\lambda}$ may be different.

First Proof. Let $T(\vec{v}) = \lambda \cdot \vec{v}$ for some $\vec{v} \neq 0$.
Then $\forall \vec{x} \in V$.
$$0 = \langle (T - \lambda I)\vec{v}, \vec{x} \rangle$$
$$= \langle \vec{v}, (T-\lambda I)^*(\vec{x}) \rangle$$
$$= \langle \vec{v}, (T^* - \bar{\lambda} I)(\vec{x}) \rangle$$

Hence $\vec{v} \perp R(T^* - \bar{\lambda} I)$

So $R(T^* - \bar{\lambda} I) \neq V$. i.e., $T^* - \lambda I$ is not onto and hence not one-to-one

$\Rightarrow$ $N(T^* - \bar{\lambda} I)$ contains at least one nonzero vector, which is an eigenvector of T^* associated with $\bar{\lambda}$.

Second proof: (of the matrix version)

A has eigenvalue λ $\iff$ $A - \lambda I$ is singular

$\iff$ $\det(A - \lambda I) = 0$

$\iff$ $\det(A^* - \bar{\lambda} I) = 0$

$\iff$ $\bar{\lambda}$ is an eigenvalue of A^*

§ Normal operator.

Def: V inner product space. $T \in \mathcal{L}(V)$

T is normal if $TT^* = T^*T$

Example / Definition.

- T is unitary (when $\mathbb{F} = \mathbb{C}$) or orthogonal (when $\mathbb{F} = \mathbb{R}$) if $TT^* = T^*T = I$.
- T is self-adjoint / Hermitian if $T^* = T$
- T is anti-self-adjoint / skew-Hermitian if $T^* = -T$.

Corresponding definition for matrices :

- A is **normal** if $A^*A = AA^*$.
- $A \in M_{n\times n}(\mathbb{C})$: A **unitary** if $A^*A = AA^* = I$

  Hermitian if $A^* = A$; **skew-Hermitian** $A^* = -A$
- $A \in M_{n\times n}(\mathbb{R})$: A **orthogonal** if $A^TA = AA^T = I$

  $(A^* = A^T)$ **symmetric** if $A^T = A$; **skew-symmetric** $A^T = -A$

Recall: $T \in \mathcal{L}(V)$ is called diagonalizable if $\exists$ basis of eigenvectors

When V is an inner product space, $T \in \mathcal{L}(V)$ is "diagonalizable"

if $\exists$ orthonormal basis of eigenvectors.

Main Theorem: Suppose V is a finite-dimensional complex inner product space $(F = \mathbb{C})$

Then T is normal $\Longleftrightarrow$ T is "diagonalizable"

i.e., $\exists$ an orthonormal basis for V consisting of eigenvectors of T.

Rmk: • $\mathbb{F}=\mathbb{C}$ is essential because char. poly needs split.

When $\mathbb{F}=\mathbb{R}$. not all normal operators are "diag".

$T: L_A = \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$T^* = L_{A^*}$ $A^* = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$\Rightarrow TT^* = T^*T = I$

T normal but not diag.

• For infinite-dim V, $\exists$ Counterexample to the Theorem [P372, Ex3]

Proof of Main Theorem is divided into 3 Steps.

Step 1: "$\Leftarrow$" if part

Step 2: Suppose $T \in \mathcal{L}(V)$. V fin-dim complex inner product space

$\exists$ orthonormal basis β s.t. $[T]_\beta$ is upper triangular.

(Here, "normal" is not needed, but $F = \mathbb{C}$ is essential)

Step 3: T normal + $[T]_\beta$ upper triangular

$\Rightarrow$ $[T]_\beta$ is diagonal

Step 1: pf of $\Leftarrow$.

Suppose β is an orthonormal basis for V of eigenvectors of T. then $[T]_\beta$ is diagonal, and $[T^*]_\beta = [T]_\beta^*$ is also diagonal.

Since diagonal matrices commute, we have

$$[TT^*]_\beta = [T]_\beta \cdot [T^*]_\beta = [T^*]_\beta [T]_\beta = [T^*T]_\beta$$

Hence $TT^* = T^*T$

$\square$

Step 2:

Theorem (Schur) Let $T \in \mathcal{L}(V)$ where V is finite-dim inner product space.

Assume further that char. poly $f_T(t)$ splits.

Then $\exists$ an orthonormal basis β for V s.t. $[T]_\beta$ is upper triangular.

pf: Induction on $n := \dim(V)$.

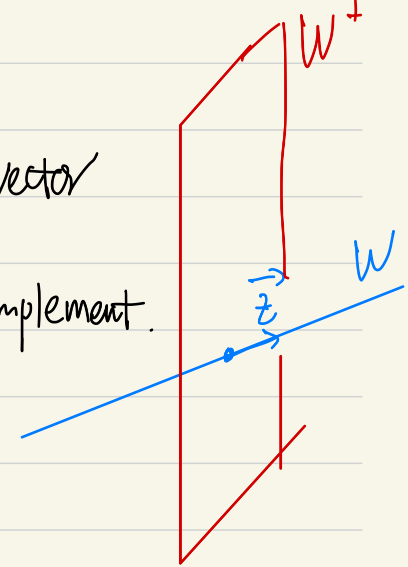
- $n=1$. trivial
- Assume true for $n-1$, to show true for n.

Since char poly $f_T(t)$ splits, T has an eigenvalue thus also eigenvector

By the earlier prop, T^* also has an eigenvector.

Assume $T^*(\vec{z}) = \lambda \vec{z}$ for some $\vec{z}$ unit eigenvector

Set $W = \text{Span}(\{\vec{z}\})$, let $W^\perp$ be the orthogonal complement.



- $W^\perp$ is T-invariant.

pf: Let $\vec{y} \in W^\perp$, to show $T(\vec{y}) \in W^\perp$

Note $\langle T(\vec{y}), \vec{z} \rangle = \langle \vec{y}, T^*(\vec{z}) \rangle = \langle \vec{y}, \lambda \cdot \vec{z} \rangle = 0$.

- In addition, $\dim W^{\perp} = n-1$.

$$T_{W^{\perp}} : W^{\perp} \longrightarrow W^{\perp}$$

- Char poly. of $T_{W^{\perp}}$ divides char. poly of T

$\Rightarrow$ $f_{T_{W^{\perp}}}(t)$ splits.

Induction hypothesis implies that $\exists$ orthonormal basis γ for $W^{\perp}$ s.t. $[T_{W^{\perp}}]_{\gamma}$ is upper triangular.

Let $\beta := \gamma \cup \{\vec{z}\}$: orthonormal basis for V.

induction hypothesis

$$[T]_\beta = \left(\begin{array}{c|c} [T(\gamma)]_\beta & T(\vec{z}) \end{array} \right) = \left(\begin{array}{c|c} \begin{matrix} [T_{W^\perp}]_\gamma & * \\ 0 & \end{matrix} & T(\vec{z}) \\ \hline 0 & \end{array} \right)$$

Since $W^\perp$ is T-inv.

upper triangular !

$\square$

Step 3: T normal, β orthonormal basis. $[T]_\beta$ upper triangular

$\Longrightarrow$ $[T]_\beta$ diagonal.

Theorem: Let $T \in \mathcal{L}(V)$ be normal. Then we have.

(a). $\|T(\vec{x})\| = \|T^*(\vec{x})\|$ $\quad \forall \vec{x} \in V$

(b). $T - cI$ is normal $\quad \forall c \in F$

(c) If $T(\vec{x}) = \lambda \vec{x}$, then $T^*(\vec{x}) = \bar{\lambda} \vec{x}$

(d). If $T(\vec{x}_1) = \lambda_1 \vec{x}_1$, $T(\vec{x}_2) = \lambda_2 \vec{x}_2$ and $\lambda_1 \neq \lambda_2$

then $\vec{x}_1$ and $\vec{x}_2$ are orthogonal.

pf:

(a). $$\|T(\vec{x})\|^2 = \langle T(\vec{x}), T(\vec{x}) \rangle$$
$$= \langle T^*T(\vec{x}), \vec{x} \rangle$$
$$= \langle TT^*(\vec{x}), \vec{x} \rangle$$
$$= \langle T^*(\vec{x}), T^*(\vec{x}) \rangle$$
$$= \|T^*(\vec{x})\|^2.$$

(b). Check $(T-cI)^* \cdot (T-cI) = (T-cI) \cdot (T-cI)^*$

Exercise: Use $(T-cI)^* = T^* - \bar{c}I$. and $TT^* = T^*T$.

(c). $(T-\lambda I)\vec{x} = 0$.

By part (b), $T-\lambda I$ is normal.

By part (a). $0 = \|(T-\lambda I)\vec{x}\| = \|(T-\lambda I)^*(\vec{x})\|$

$\|(T^*-\bar{\lambda} I)\vec{x}\|$

Hence $(T^*-\bar{\lambda} I)(\vec{x}) = 0$

$\iff T^*\vec{x} = \bar{\lambda}\cdot\vec{x}$.

(d). $\langle T(\vec{x}_1), \vec{x}_2\rangle = \langle \lambda_1\vec{x}_1, \vec{x}_2\rangle = \lambda_1\langle\vec{x}_1, \vec{x}_2\rangle$

$= \langle \vec{x}_1, T^*(\vec{x}_2)\rangle \overset{\text{part (c)}}{=} \langle \vec{x}_1, \bar{\lambda}_2\vec{x}_2\rangle = \lambda_2\langle\vec{x}_1, \vec{x}_2\rangle$

Since $\lambda_1 \neq \lambda_2$, $\langle\vec{x}_1, \vec{x}_2\rangle = 0$.

$\square$

Pf of Step 3: Suppose $\beta = \{\vec{v_1}, \vec{v_2} \dots \vec{v_n}\}$ orthonormal basis

$$[T]_\beta = \begin{array}{c} \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ & \lambda_2 & 0 & 0 \\ & & \lambda_3 & 0 \\ & & & \ddots \end{pmatrix} \\ T(\vec{v_1}) \end{array} \quad \text{upper triangular.}$$

Note that $\vec{v_1}$ is an eigenvector of T.

Suppose that $\vec{v_1}, \dots, \vec{v_{k-1}}$ are eigenvectors, we prove $\vec{v_k}$ is also an eigenvector.

(Then by induction, all vectors in β are eigenvectors)

Let $A := [T]_\beta$ upper triangular.

$$T(\vec{v_i}) = \lambda_i \vec{v_i} \quad \forall\ 1 \le i \le k-1 \quad \left(\Rightarrow T^*(\vec{v_i}) = \bar{\lambda}_i \vec{v_i} \right).$$

$$T(\vec{v_k}) = \underbrace{A_{1k}}_{= 0\,?} \vec{v_1} + \underbrace{A_{2k}}_{= 0\,?} \vec{v_2} + \cdots + A_{kk} \vec{v_k}.$$

For $1 \le i \le k-1$,

$$\begin{aligned} A_{ik} &= \langle T(\vec{v_k}), \vec{v_i} \rangle \\ &= \langle \vec{v_k}, T^*(\vec{v_i}) \rangle \\ &= \langle \vec{v_k}, \bar{\lambda}_i \vec{v_i} \rangle \\ &= 0. \end{aligned}$$

Hence $T(\vec{v_k}) = A_{kk} \vec{v_k}$ $\Rightarrow$ $\vec{v_k}$ is an eigenvector.

$\Rightarrow$ $[T]_\beta$ is diagonal $\square$

§ Analogous result in real case.

Main Theorem: Let $T \in \mathcal{L}(V)$, where V is a real, finite-dimensional

Then: T self adjoint $\Longleftrightarrow$ T "diagonalizable"

$(T = T^*)$

Rmk: In matrix vertion. T self-adjoint $\quad F = \mathbb{R}$.

$\Longleftrightarrow [T]_\beta$ symmetric matrix

Pf of $\Leftarrow$: Assume there is an orthonormal basis β of eigenvectors.

then $[T]_\beta$ is a diagonal matrix with real entries,

$$\Rightarrow \quad [T^*]_\beta = [T]_\beta{}^* = [T]_\beta . \quad \Leftrightarrow \quad T^* = T$$

$\square$

To prove $\Rightarrow$, need splitting char. poly (in order to apply Schur Thm)

Lemma: T self adjoint operator on fin-dim inner product space V. Then

$F = \mathbb{C}$ or $\mathbb{R}$

(a). All eigenvalues of T are real.

(b). Characteristic polynomial $f_T(t)$ splits.

Proof: (a) Suppose $T(\vec{x}) = \lambda \vec{x}$ for $\vec{x} \neq 0$.

Then $T^*(\vec{x}) = \bar{\lambda} \vec{x}$ since T is normal.

Hence $\lambda \vec{x} = T(\vec{x}) = T^*(\vec{x}) = \bar{\lambda} \vec{x}$.

$\Rightarrow \lambda = \bar{\lambda}$: λ is real.

(b). Suppose $\dim V = n$. β orthonormal basis for V. $[T]_\beta = A$

Consider $L_A . \mathbb{C}^n \longrightarrow \mathbb{C}^n$. L_A is self-adjoint.

By Fundamental Thm of algebra, $f_{L_A}(t)$ splits (over $\mathbb{C}$) into factor of the form $(t-\lambda_1)\cdots(t-\lambda_n)$, where λ_i's are eigenvalues.

By (a), all eigenvalues are real

$\Rightarrow f_{L_A}(t)$ also splits over $\mathbb{R}$.

As $f_T(t) = f_{L_A}(t)$, it also splits

$\square$

Pf of Main Thm (Self adjoint $\Rightarrow$ "diagonalizable" for $F=\mathbb{R}$)

(Step 2:) The above lemma implies $f_T(t)$ splits over $\mathbb{R}$.

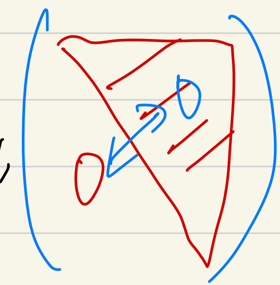
By Schur's Theorem, $\exists$ an orthonormal basis β for V s.t. $[T]_\beta =: A$ is upper triangular

(Simpler Step 3:)

$T=T^*$

Note: $A^* = ([T]_\beta)^* = [T^*]_\beta = [T]_\beta = A$.

$\Rightarrow$ A is real symmetric, but also upper triangular,



$\Rightarrow$ A is diagonal $\square$